Mathematics

FRACTAL GEOMETRY : AN INTRODUCTION

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ABSTRACT

This paper aims at providing basic concept of Fractal Geometry, its origin and development, and how the use of fractals proliferated in modern science, art and technology, ushering paradigmatic change in our thinking. Fractals are the geometric shapes with fractional dimension. Most of the physical phenomena are dynamical system where an initial fluctuation can cause tremendous effect. Chaos and coherence are built in dynamical systems and only through the development of fractal geometry, the complexity of phenomena is getting unraveled at an amazing speed.

Keywords: Attractor, Cantor set, Euclid, Fractal, Mandelbrot

Introduction

Geometry is the branch of mathematics which is concerned with the points, lines, curves and surfaces. During 3rd century B.C., by assuming a small set of intuitively appealing axioms and deducing many other propositions from those, Euclid put Geometry into axiomatic form; which is commonly known as Euclidean Geometry. For many centuries, Euclidean Geometry served as an important tool in solving the geometrical and astronomical problems. However, the Euclidean Geometry is not capable of studying irregular and fragmented patterns around us. Benoit B. Mandelbrot, the father of Fractal Geometry describes the reason to transcend Euclidean Geometry, "*Why is geometry often described as "cold" and "dry"? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline or a tree. Clouds are not sphere, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in straight line."(Mandelbrot,* 1982) The study of these irregular patterns is out of the purview of classical geometry, and these were set aside by Euclid as being "formless". In 20th Century, Mandelbrot introduced a new Geometry which is able to describe the shapes of the irregular and fragmented patterns around us, known as Fractal Geometry. Fractal Geometry brings together a large class of objects, under one roof, and it separates the classical mathematics of 19th century from the modern mathematics of 20th century. Classical mathematics is rooted around the regular geometric structures of Euclid and the dynamics of Newton, whereas the modern mathematics commences with the Cantor's set theory and Peano's space-filling curve.

Topology is a well defined branch of mathematics. Though it is a major area of mathematics, it is not capable of explaining the theory of Form. Topology is the study of qualitative properties of certain objects that are invariant under certain kind of transformations, especially those properties that are invariant under a certain kind of equivalence. In topology, all single island coastlines are of the same form, because they are topologically identical to a circle. Similarly, all pots with two handles are of the same form, topologically, because, if both are infinitely flexible and compressible, these can be molded into any other continuously, without tearing any new opening or closing up and old one. To differentiate the form of topologically identical objects, Mandelbrot went beyond topology and proposed fractal dimension.

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In mathematics, the dimension is generally defined as the minimum number of coordinates required to specify each point within a space or object. Generalizing the concept of dimension, Hausdorff and Besicovitch gave the notion of Hausdorff-Besicovitch dimension. We denote it by D. However, the concepts of irregularity or fragmentation cannot be made by defining dimensions as a number of coordinates. Euclidean Geometry is limited to sets for which all the useful dimensions coincide, so Mandelbrot termed these dimensionally concordant sets. The concept of dimension is not restricted to physical objects. In the study of Fractal Geometry, the dimensions fail to coincide; therefore, these sets are called *dimensionally discordant*.

In the early 20th century *Karl Menger, L E J Brouwer, Pavel Urysohn* and *Henri Lebesgue* introduced the idea of topological dimension. We denote it by DT. While working in Euclidean space En, both DT and D are at least 0 and at most n. According to Mandelbrot, "*A fractal is by definition a set for which the Hausdorff Besicovitch dimension strictly exceeds Topological dimension*." The dimension DT is always an integer, but D need not be an integer and the two dimensions need not coincide. Instead they satisfy the inequality, D ≥ DT. In the Euclidean Geometry, D = DT. However, in case of Fractal Geometry, D > DT.

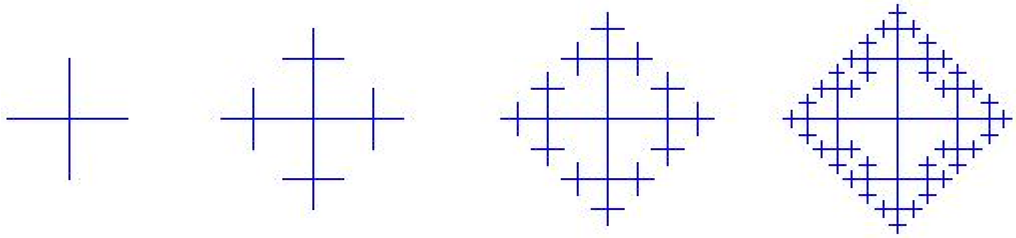
Every set with a non-integer D is a fractal. For example original Cantor Set is a fractal because,

$$D = \frac{\log 2}{\log 3} \sim 0.6309 > 0, \textit{while } D_T = \mathbf{0}$$

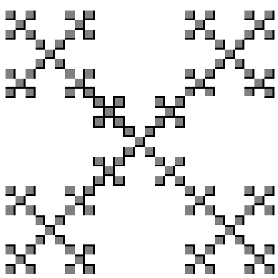
A fractal is "*a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole*" [1]. The term fractal was introduced by *Benoît Mandelbrot* in 1975. It was derived from the Latin fractus meaning "broken" or "fractured." A fractal is irregularly shaped and it cannot be described by using the traditional aspects of Euclidean geometry. A fractal is a quantity or objet which exhibits self-similarity on all scales. Even at arbitrarily small scales, it has fine structures. Fractals are self-similar, meaning thereby that a fractal is exactly or approximately similar to a part of itself. Self-similarity means that each small portion, when magnified, can reproduce exactly a larger portion. Mathematically, a fractal is based on some equation which undergoes iteration. The length of a coastline measured with different length rulers may be an example of fractal. The shorter the ruler, the longer the length measured, a paradox known as the coastline paradox (for details we refer [1]). Examples of fractals among natural objects include clouds, mountain ranges, lightning bolts, snow-flakes, various vegetables (e.g., cauliflower and broccoli), and animal coloration patterns. These are self-similar to certain degree. However, all self-similar objects are not fractals. For example, the real line is formally self-similar but fails to have other fractal characteristics; for instance, it is regular enough to be described in Euclidean terms.

Mathematically, the Cantor set, Mandelbrot set, Julia set, Sierpinski triangle, Sierpinski carpet, space-filling curve, Koch curve, Menger sponge and dragon curve are considered as example of fractals. Chaotic dynamical systems are also associated with fractals. The easiest fractals are those based on iterated function system. Cantor set, Sierpinski carpet, Peano curve, Menger sponge are some examples of such fractals. These fractals have a fixed geometric replacement rule. Some fractals are generated by stochastic process rather than the deterministic process. These are called random fractals. Some examples are trajectories of Brownian motion, fractal landscape etc. There are Escape-time fractals or orbits fractals which are defined by a rule or recurrence relation at each point in a space. Examples of this type are the Mandelbrot set and Julia set. Another type of fractals is known as strange attractors, which are generated by iteration of a map or the solution of a system of initial-value differential equations that exhibit chaos.

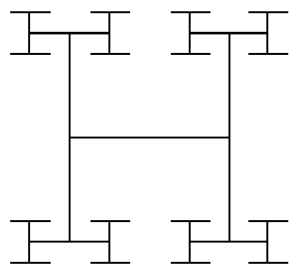
Fractals are geometrical figures that are generated by starting with a very simple pattern that grows through the application of certain rule or rules. In many cases, the rules to make the figure grow from one stage to the next that involves taking the original figure and modifying it or adding to it. This process can be repeated recursively (the same way over and over again) an infinite number of times. For example, if we start with a + sign and grow it by adding a half size + in each of the four line ends. We repeat the exact same process recursively as many times as we desire. We get a fractal as shown in the figure below and call this the Plusses fractal:



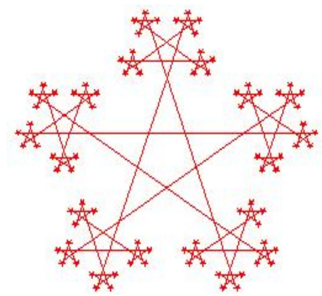
Following figures shows some fractals which can be drawn easily:



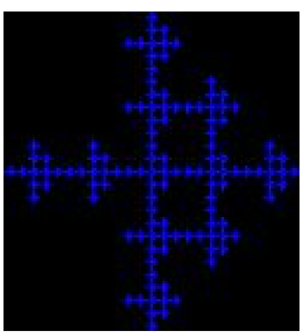
Box - Fractal



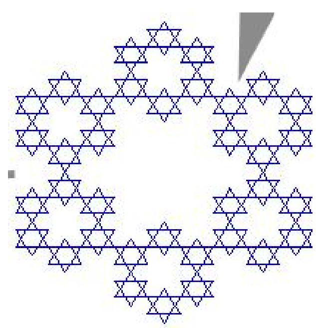
H - Fractal



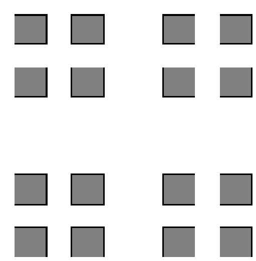
Star - Fractal



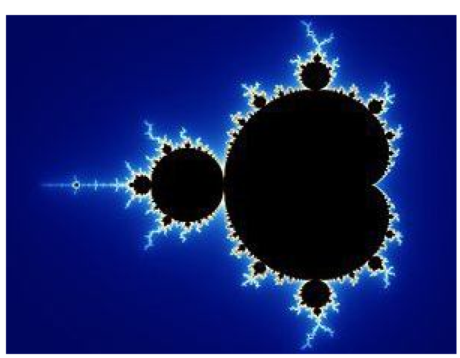
Arrow - Fractal



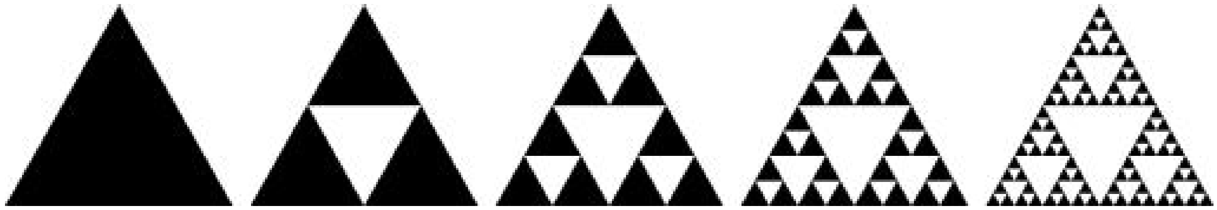
Star of David



Cantor Square Fractal

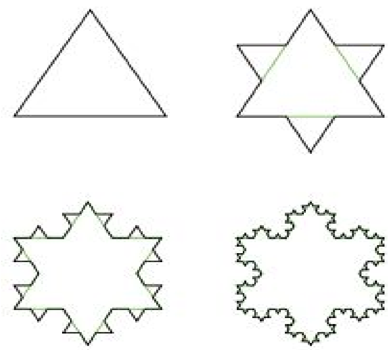


Mandelbrot Set



The Sierpinski triangle is a fractal named after the Polish mathematician Wacław Sierpiński who described it in 1915.

The Koch snowflake is one of the earliest fractal curves, which were described by Helge Von Koch, Swedish mathematician.



Koch Snoflakes

Fractals are very useful. Computer graphics have been one of the earliest applications of fractals. Fractals can achieve realism, beauty and require very small storage space. A fractal "footprint" can be used to identify man made versus natural features on aerial mappings and tracking submarines. Scientists use fractal geometry to locate oil, identify geologic faults, and possibly predicting earthquakes. Statistical models using fractal geometry are used to test for stress loading on oil rigs and turbulence effects on aircraft. Acid rain and corrosion can be modeled using fractal geometry. Our universe is self-similar to much extent and it can be studied on the basis of cluster fractals. Even the big bang theory and understanding of the structure of the universe can be improved with fractals. Edward Nortan Lorenz has discovered that weather, which behaves very chaotically, creates fractal patterns. On the basis of his work, he gave the notion of *'strange attractor'* and coined the term '*butterfly effect*'. One of the very odd applications of fractals is in turning geometric patterns in sound patterns. Fractals are used in movies for landscapes, dinosaur skin textures, etc. For instance, the raindrops on the skin of the dinosaurs in 'Jurassic Park' were done using a fractal model. Fractals were marvelously used in movies 'Star Trek II'. The Wrath of Khan' and 'Return of Jedi'. This diverted the attention of many artists and producers of scientific fiction movies to the use of fractals. Mandelbrot applied fractals in information theory, economics, cosmology and fluid dynamics. He found that price changes in financial markets do not follow a Gaussian distribution, but Levy stable distributions having theoretically infinite variance. Fractals are also prevalent in art, architecture, textiles and sculpture. By using a technique, known as *Decalcomania*, artists can produce fractal like patterns. The spring industry uses fractal geometry to test spring wire in 3 minutes instead of 3 days. Fractals are being used in the study of chemical reactions, human anatomy, plants, bacteria cultures, molecules etc. Fractal Geometry has very wide and far reaching applications in medical science. It is used in the study and treatment of lungs, AIDS, cancer, bone fractures and heartbeats etc.

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